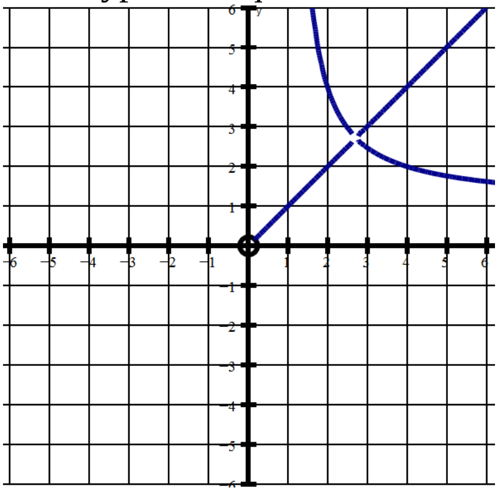
THE PERPLEXING PROBLEM OF $y^x = x^y$

If we type this equation into the Autograph program we get the following graph:



However this is not the whole story!
If we examine the part of the graph which is the same as the line $y = x$ we can see that if we let $y = x = b$ then obviously b^b **always** equals b^b

$y^x = x^y$
$1^1 = 1^1$
$2^2 = 2^2$
$3^3 = 3^3$

ALSO:

$(\frac{1}{2})^{\frac{1}{2}} = (\frac{1}{2})^{\frac{1}{2}}$
$(\frac{1}{3})^{\frac{1}{3}} = (\frac{1}{3})^{\frac{1}{3}}$
$(\frac{5}{8})^{\frac{5}{8}} = (\frac{5}{8})^{\frac{5}{8}}$

But the negative numbers also fit the equation $y^x = x^y$

Suppose $y = x = -1$ then $(-1)^{-1} = \left(\frac{1}{-1}\right)^{+1} = -1$

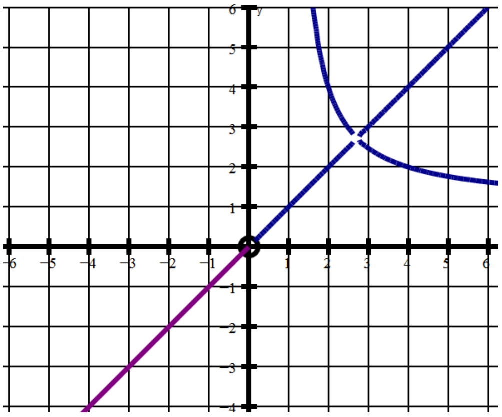
Also, if $y = x = -2$ then $(-2)^{-2} = \left(\frac{1}{-2}\right)^{+2} = \frac{+1}{4}$

Also, if $y = x = -3$ then $(-3)^{-3} = \left(\frac{1}{-3}\right)^{+3} = -\frac{1}{27}$

Also, if $y = x = -\frac{1}{2}$ then $= \left(\frac{1}{-2}\right)^{-\frac{1}{2}} = (-2)^{\frac{1}{2}} = i\sqrt{2}$

It does not matter that this is an imaginary number!
All that matters is that $y^x = x^y$ and in this case $= i\sqrt{2}$.

This means we should extend the above graph as follows: **(purple)**



The case of $y = x = 0$ is a concern of course.

Mathematicians always "shy away" from things like this with a glib comment such as "this is not defined".

I think that this is fine in some cases like $\frac{0}{0}$ which we say is "indeterminate".

I like this explanation:

If $a \times b = c \times d$ ***then*** $\frac{a}{c} = \frac{d}{b}$

Suppose $a = 6$, $b = 0$, $c = 7$ ***and*** $d = 0$

So if $6 \times 0 = 7 \times 0$ then $\frac{6}{7} = \frac{0}{0}$

In other words $\frac{0}{0}$ ***can equal ANYTHING!***

(Just substitute any numbers for a and c)

It is "indeterminate" as it is, BUT we can "determine" it.

eg $\lim_{h\to 0} \frac{2xh + h^2}{h} = \frac{0}{0}$ ***if we just let*** $h = 0$

BUT if we simplify it first:

$\lim_{h\to 0} \frac{h(2x + h)}{h}$ ***(here we can cancel the h's because h is just a small value)***

$= \lim_{h\to 0} (2x + h)$

$= 2x$

HOWEVER I think 0^0 is a bit different.

Consider $\lim_{b\to 0} b^0$

Obviously $(0.000000001)^0 = 1$ so $\lim_{b\to 0} b^0 \to 1$

Compare with $\lim_{b\to 0} 0^b$

Obviously $0^{0.000000001} = 0$ so $\lim_{b\to 0} 0^b \to 0$

I think that 0^0 can only be 0 or 1 and in this case I believe the sensible conclusion is that $0^0 \to 0$ thus completing the line $y = x$.

So instead of saying 0^0 is "NOT DEFINED" it seems sensible to simply "DEFINE" it as being equal to 0 in this particular case.

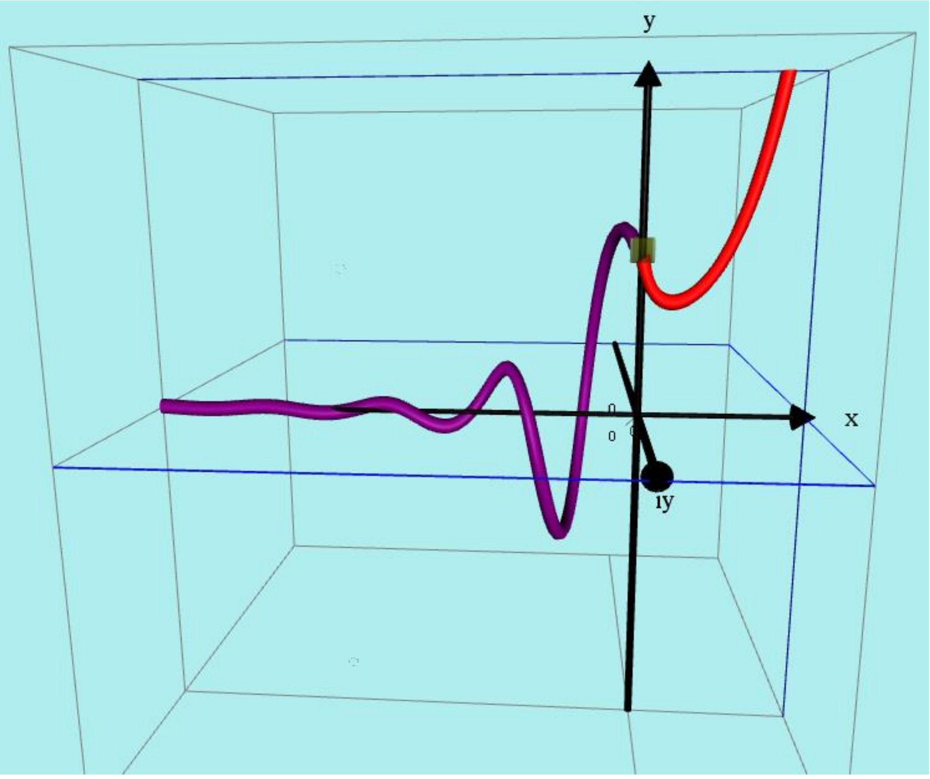
(But any purist is welcome to exclude this point if desired.)

But if $x = y = 0$ then whatever 0^0 equals (either 0 or 1) then y^x still equals x^y.

Incidentally for the graph of $y = x^x$ we have the same problem when $x = 0$ because $y = 0^0$

SEE http://screencast.com/t/m4fmGwmkrT9

The full graph of $y = x^x$ for REAL values of x (but allowing imaginary y values) is below:



Clearly** $y = \lim_{x \to 0} x^x$ **approaches** $y = 1$ **from <u>the left and from the right</u>.

So instead of saying 0^0 is "NOT DEFINED" it seems sensible to simply "DEFINE IT" as being equal to 1 in this particular case.

The most interesting types of points on $y^x = x^y$ are those like $(2, 4)$ and $(4, 2)$ because $y^x = 4^2 = 16$ and $x^y = 2^4 = 16$

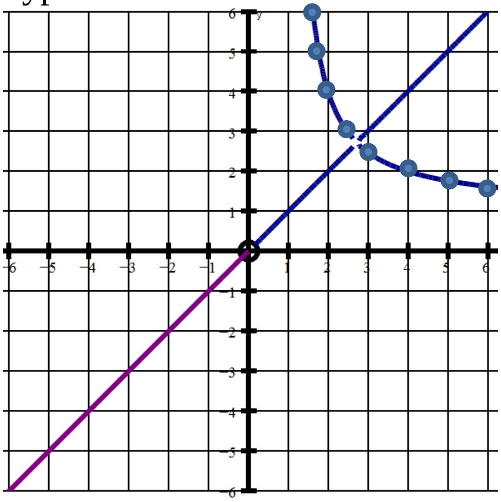
Suppose we choose $y = 5$, then we need to solve $5^x = x^5$ to find the x value. Either solving ***graphically*** by finding the intersection of $Y = 5^x$ and $Y = x^5$ or using the ***equation solver*** on a graphics calculator, we get: $x = 1.764921915$

TESTING: $5^{1.764921915} = 17.1248777$
and $1.764921915^5 = 17.1248777$

Suppose we choose $y = 6$, then solving $6^x = x^6$ we get $x = 1.624243846$

TESTING: $6^{1.624243846} = 18.36146714$
and $1.624243846^6 = 18.36146714$

Choosing $y = 3$ we get $x \approx 2.478$ so we can plot (2.478, 3) and (3, 2.478)
Points like the above examples, produce the part of the curve which resembles a hyperbola.



The apparent "hole" at $x = 2.719$ is very unusual but on solving
$2.719^x = x^{2.719}$
we do get $x = 2.719$

This means the point $x = 2.719$, $y = 2.719$ satisfies $y^x = x^y$ and this must be the point where the two sections of the graph intersect.

So I will fill in the "hole" at (2.719, 2.719)

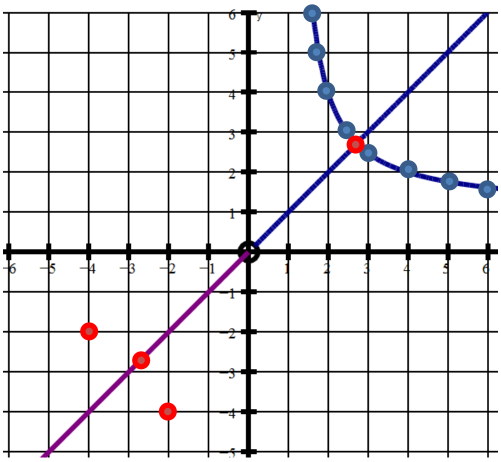
It occurred to me that I should also try $x = -2, y = -4$

TESTING: $(-2)^{-4} = (-\tfrac{1}{2})^4 = \frac{1}{16}$

$(-4)^{-2} = (-\tfrac{1}{4})^2 = \frac{1}{16}$

Also, considering $x = y = -2.718$
Obviously $(-2.718)^{-2.718} = (-2.718)^{-2.718}$
The fact that $(-2.718)^{-2.718} = -0.04177 - 0.05114i$ which is a complex number, does not matter as long as it fits $y^x = x^y$

So we can put the points (-2, -4) and (-4, -2) and (-2.718, -2.718) on the graph.
See below:



HOWEVER, trying $x = -1.76492$ ***and*** $y = -5$

$(-1.76492)^{-5} = -0.05839457758$ **BUT** $(-5)^{-1.76492} = 0.04318 + i\, 0.03931$

Disappointingly, this does not fit the equation $y^x = x^y$ **because** $y^x \neq x^y$

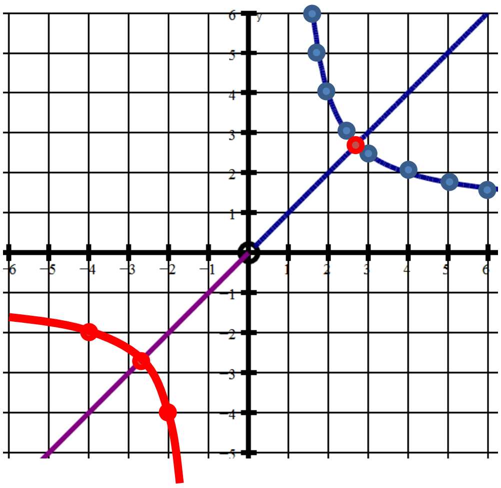
Similarly, trying $x = -2.478$ ***and*** $y = -3$

$(-2.478)^{-3} = -0.0657$ **BUT** $(-3)^{-2.478} = 0.00454 - i\, 0.0656$

Also this does not fit the equation $y^x = x^y$ because $y^x \neq x^y$

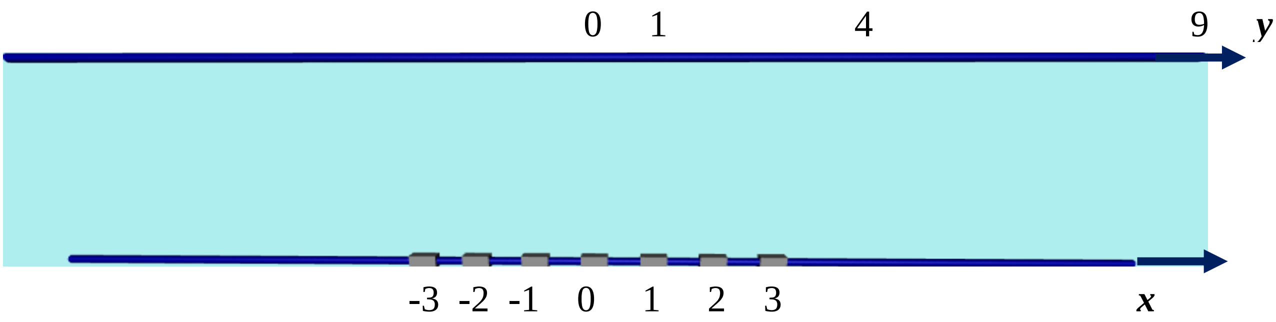
Obviously, I was <u>hoping</u> that the part of the curve resembling a hyperbola would be "reflected" or "rotated" to join up the points (-2, -4) and (-4, -2) and (-2.718, -2.718) in the 3rd **quadrant.**

See **RED CURVE** below:

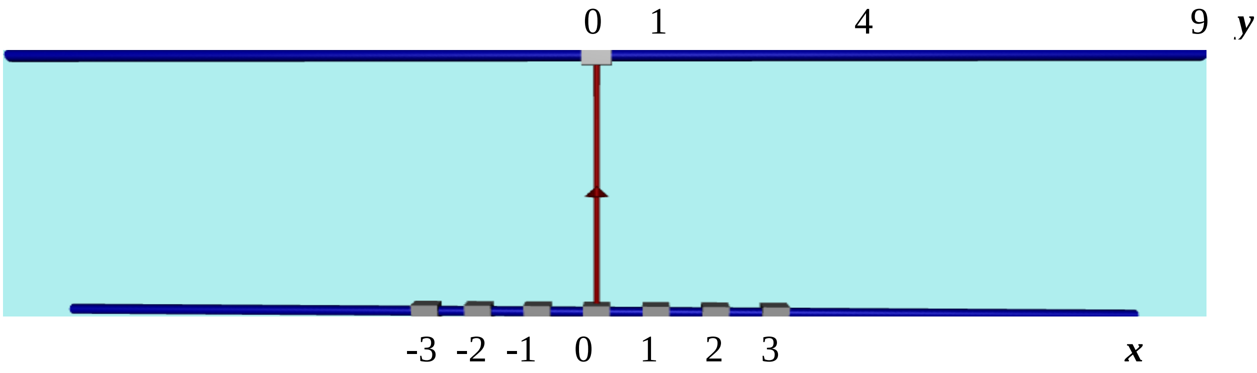


The following is a slight diversion but it does apply to this problem:

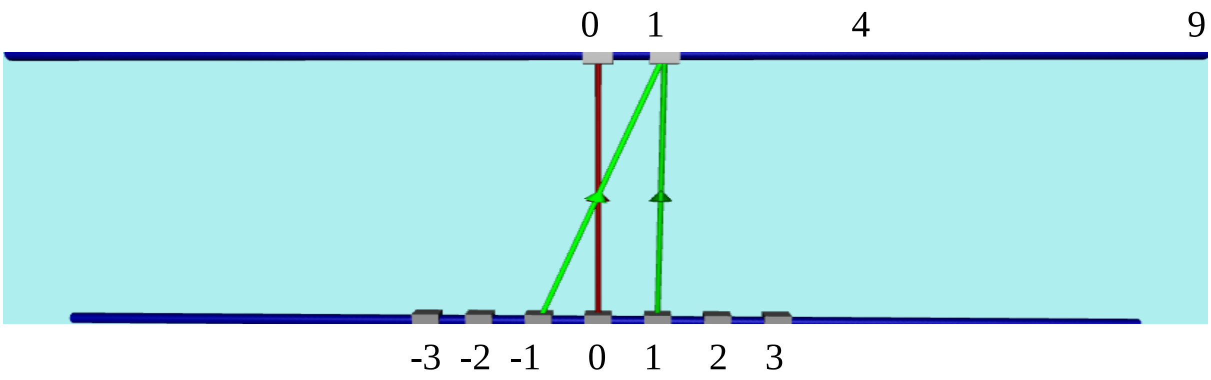
Think of $y = x^2$ *as a process of* ***MAPPING x*** *values from an* ***x axis*** *onto* ***y*** *values* on a ***y axis*** as shown below:



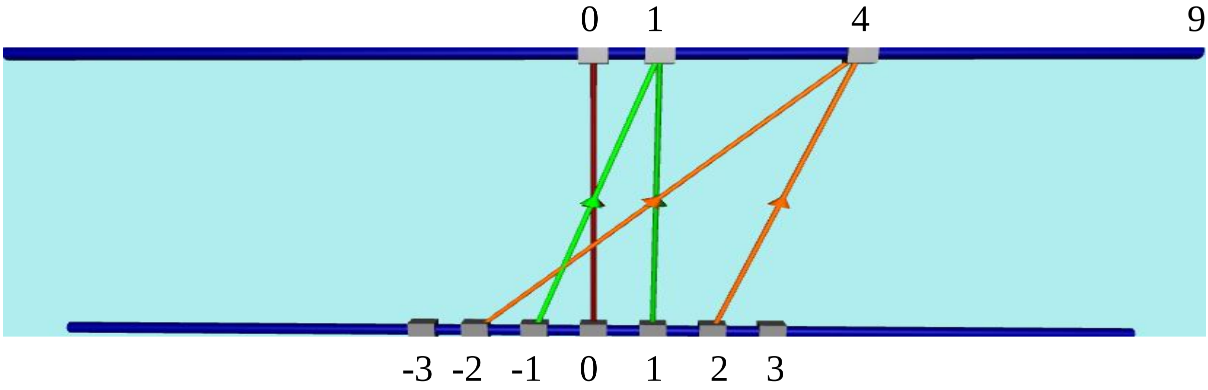
Firstly $0^2 = 0$ so we join $x = 0$ to $y = 0$



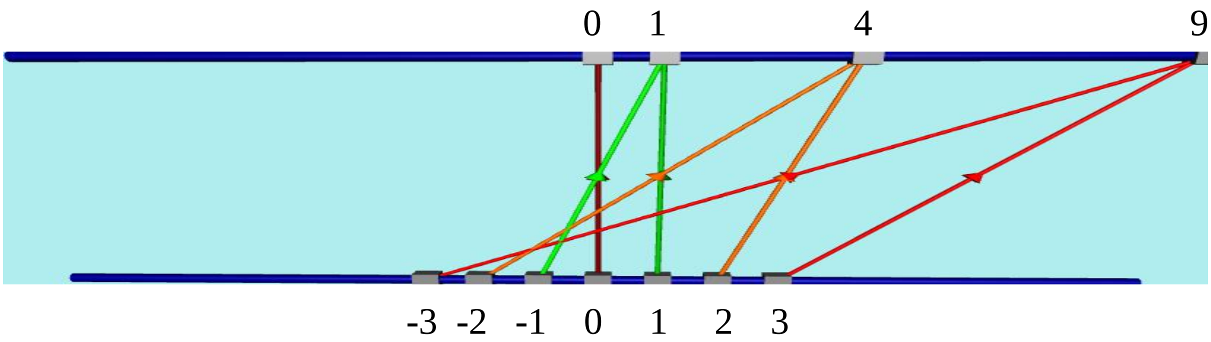
Now if $x = \pm 1$, $y = +1$



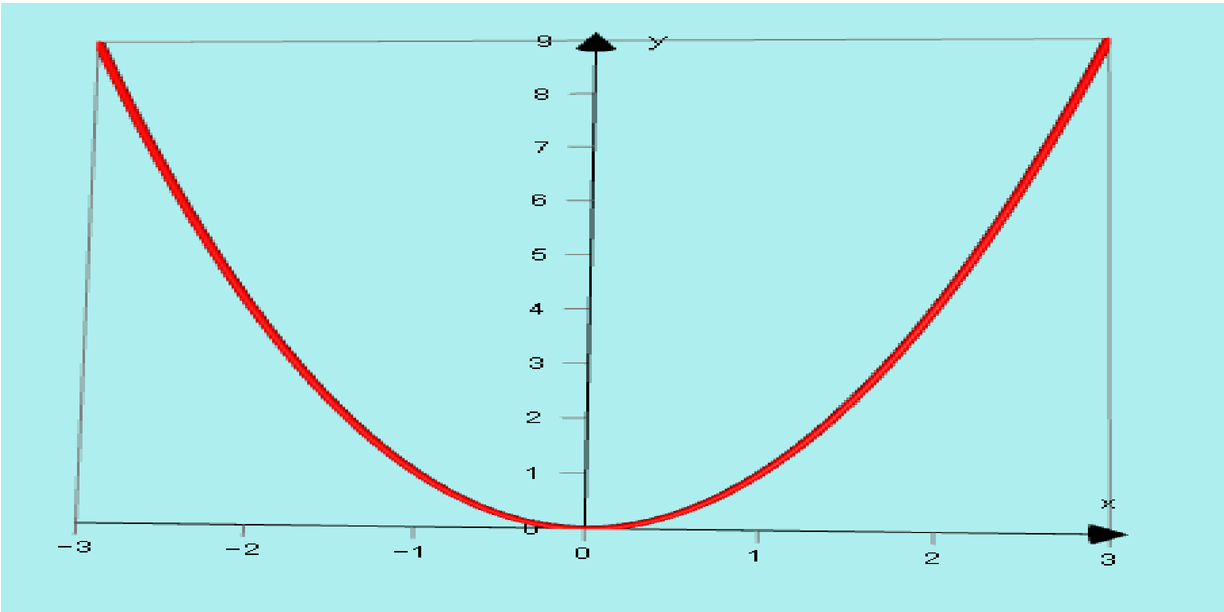
If x = ±2, y = +4



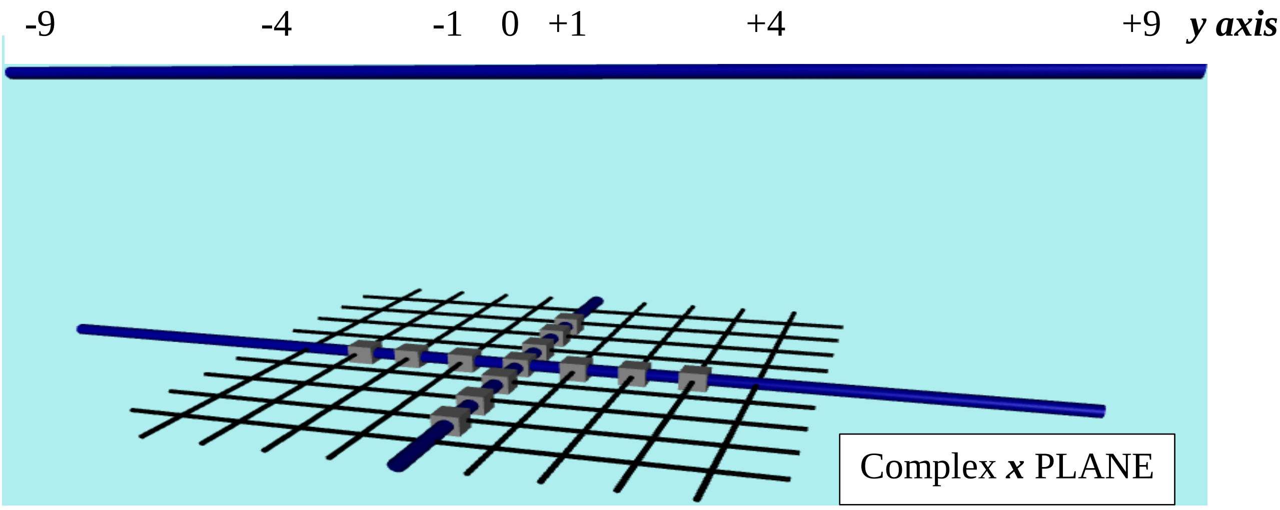
And if x = ±3, y = +9



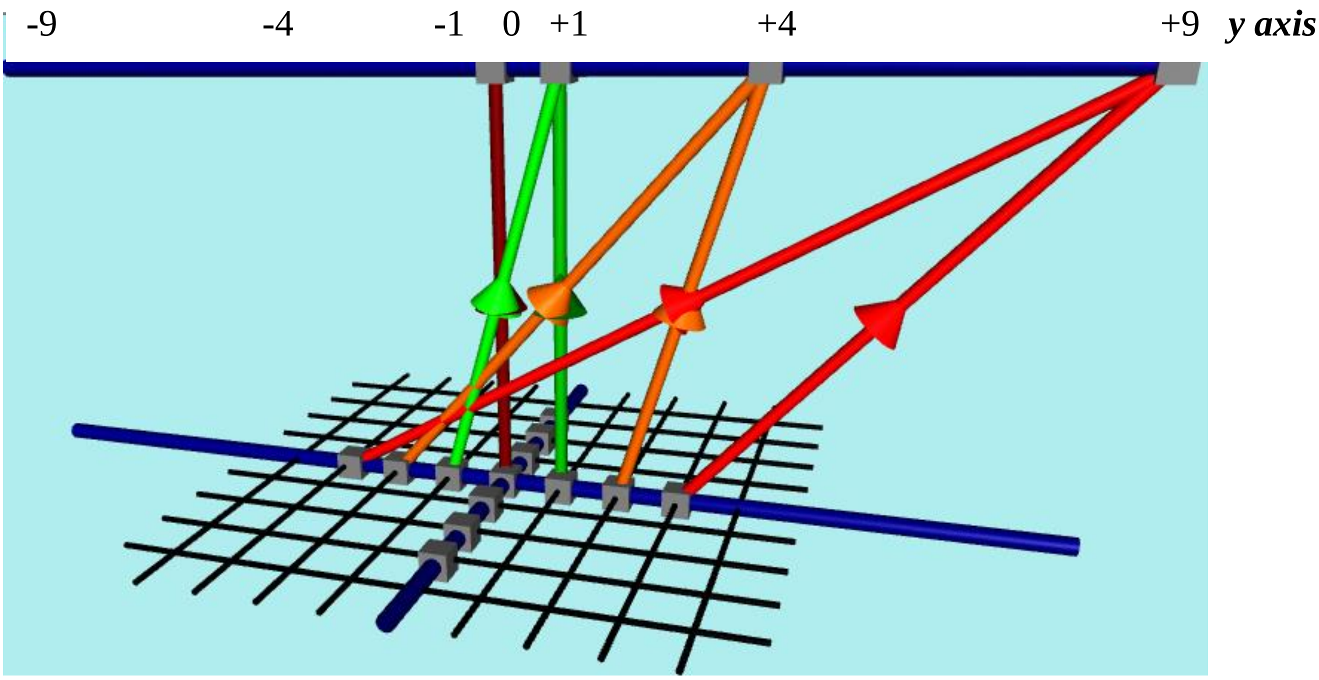
This of course produces the "normal" parabola $y = x^2$.



But now let us repeat this process of MAPPING ***x*** values from an ***x PLANE*** onto ***y*** values on a ***y axis*** as shown below:

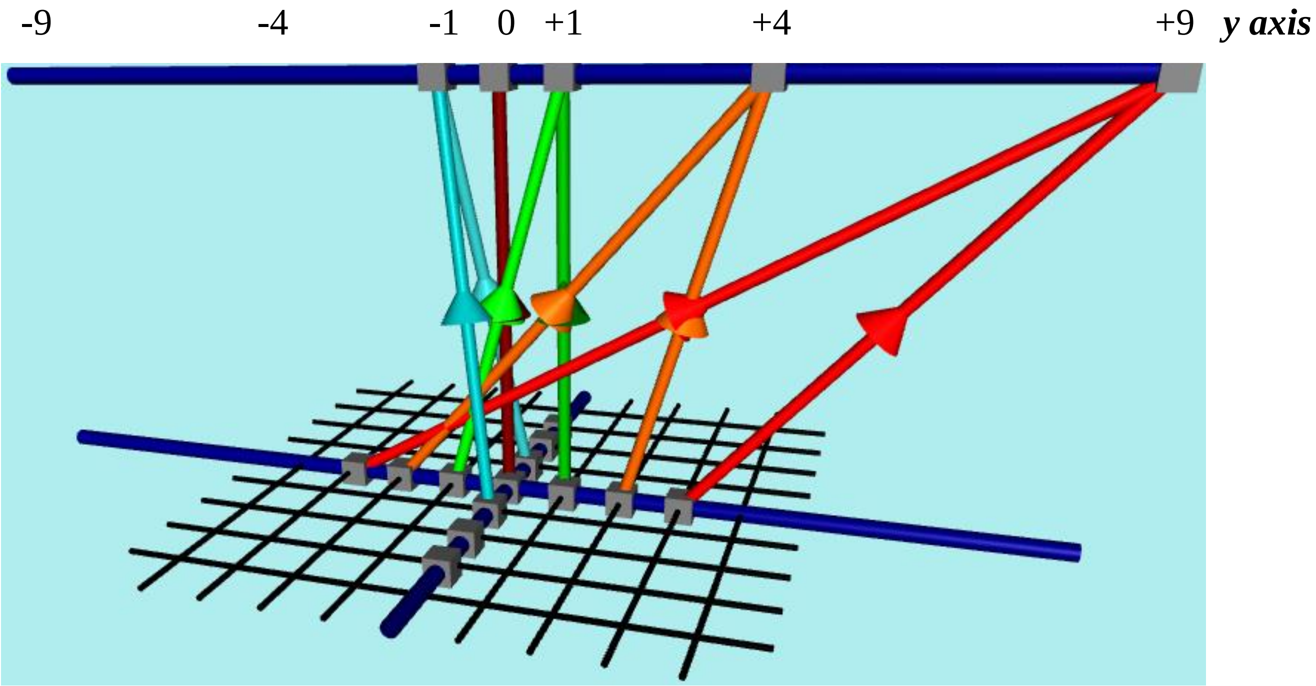


The diagram below shows ***x = ±3, y = +9 (red), x = ±2, y = +4 (orange)***
x = ±1, y = +1 (green) and x = 0, y = 0 (brown)

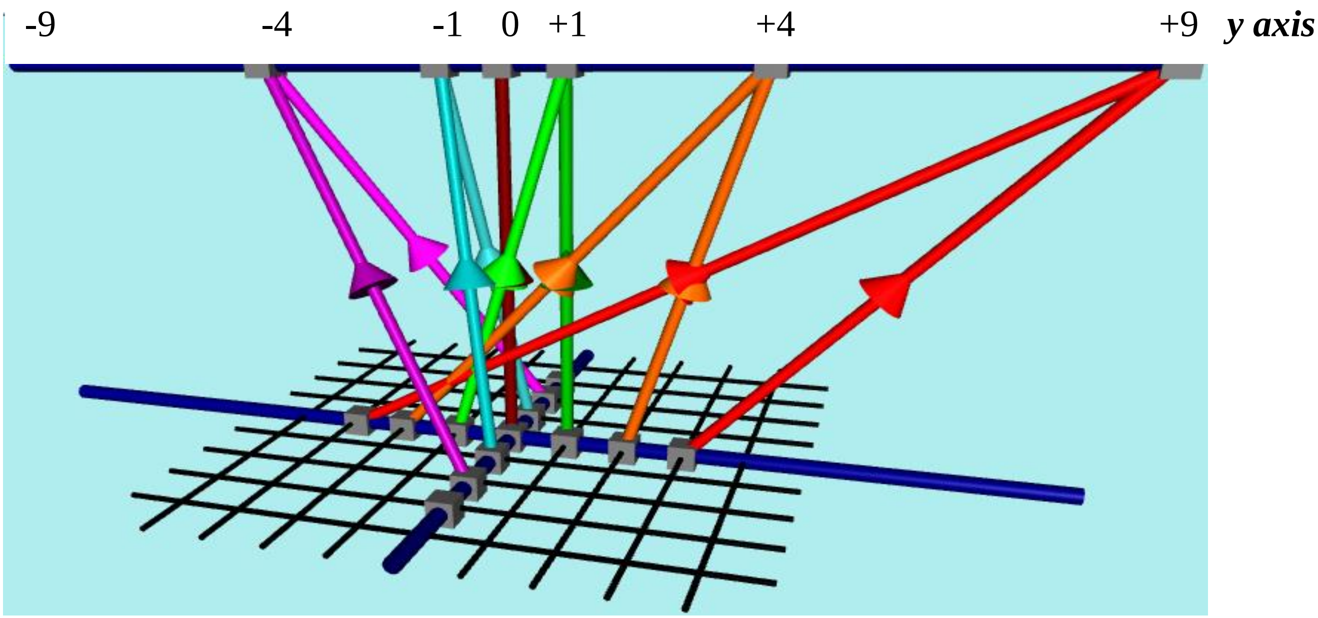


BUT NOW WE CAN ADD SOME IMAGINARY x VALUES WHICH PRODUCE REAL y VALUES. (This is the whole idea of Phantom Graphs!)

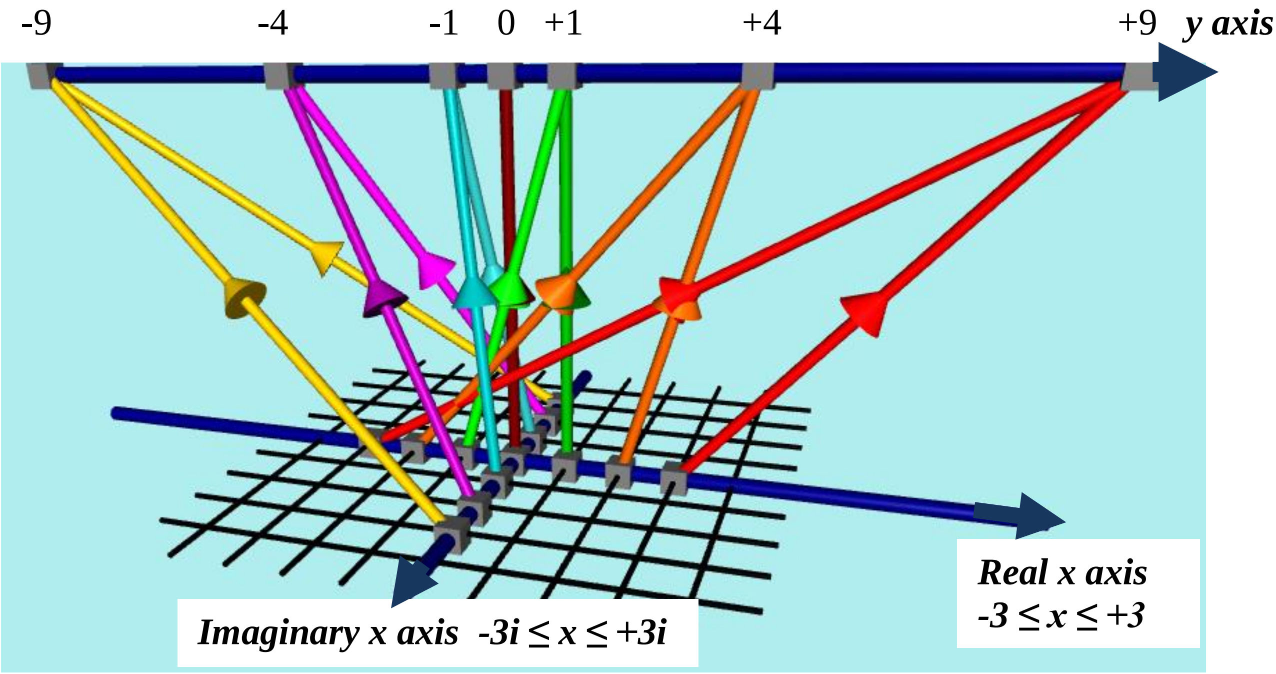
Here we add $x = \pm i$ which map onto $y = -1$ (turquoise)



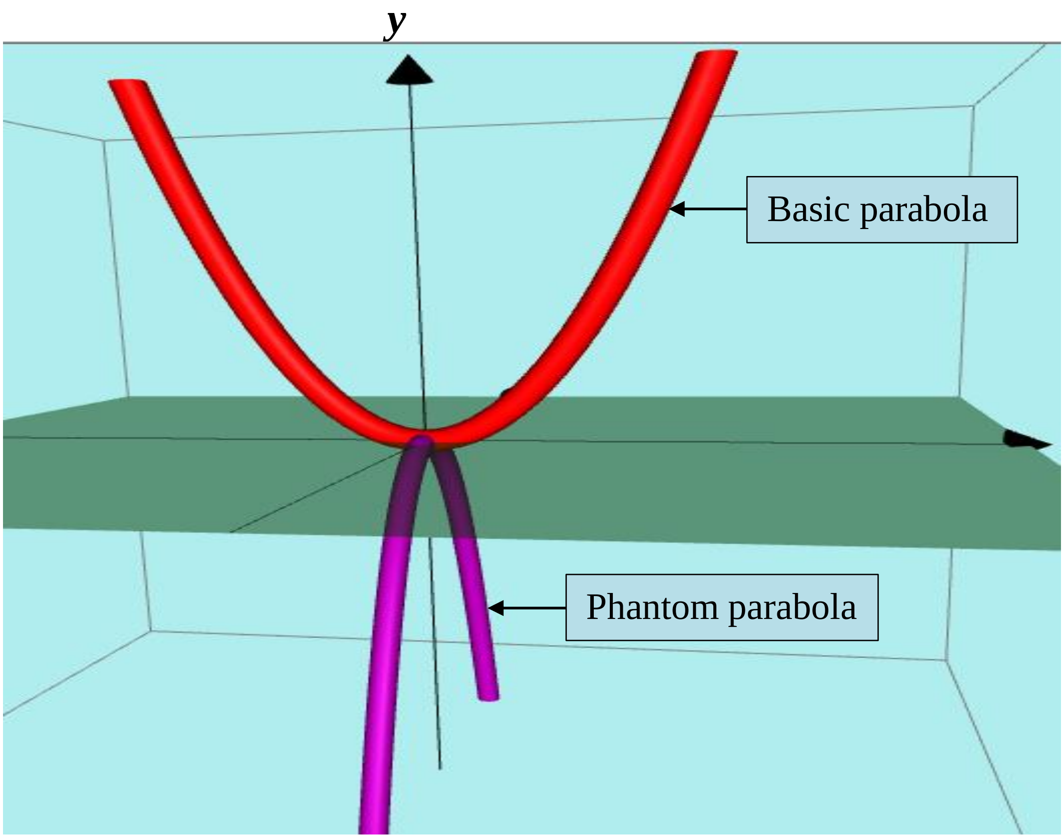
Now we add $x = \pm 2i$ and $y = -4$ (purple)



And finally ***x* = ±3*i* and *y* = – 9** *(yellow)*

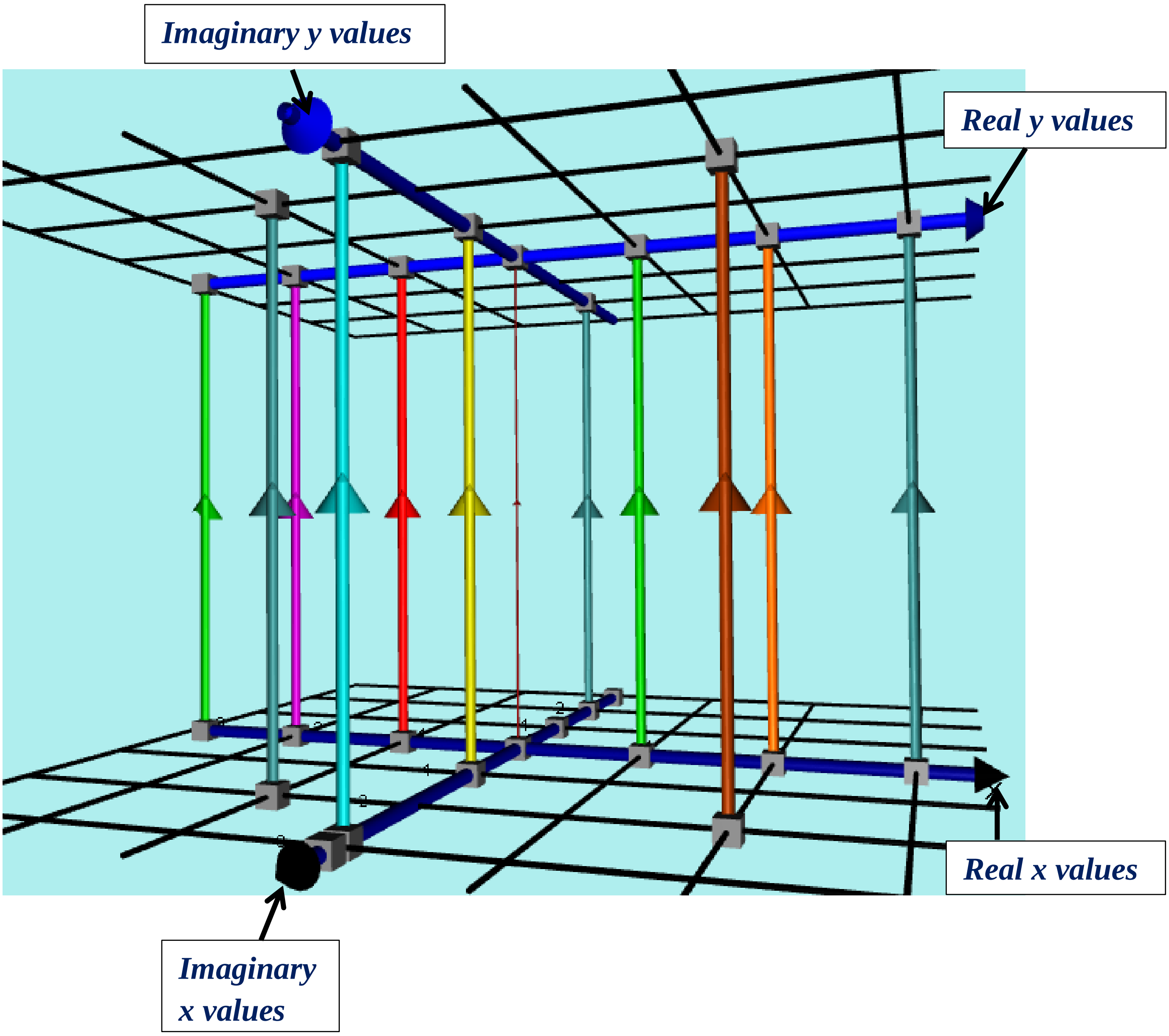


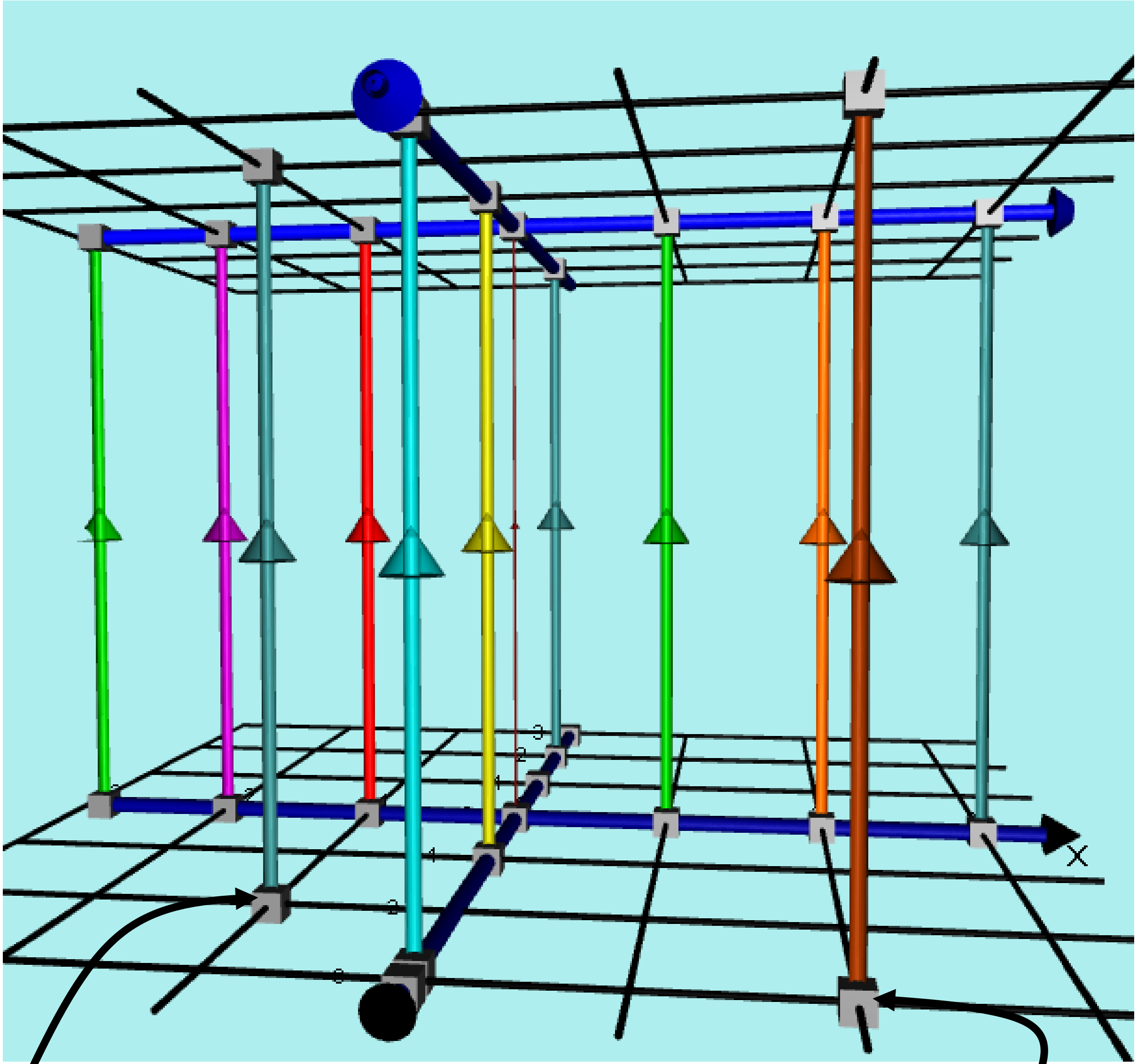
This of course produces the basic PHANTOM GRAPH of y = x^2 if we use the complex x plane and place the vertical y axis through it.



The whole point in the last 5 pages was to use the idea of mapping *complex x values* onto *complex y values* for the problem $y^x = x^y$

I established earlier that ALL real or imaginary values such as $x = y = a + ib$ must satisfy $y^x = x^y$ so the following diagram indicates this.





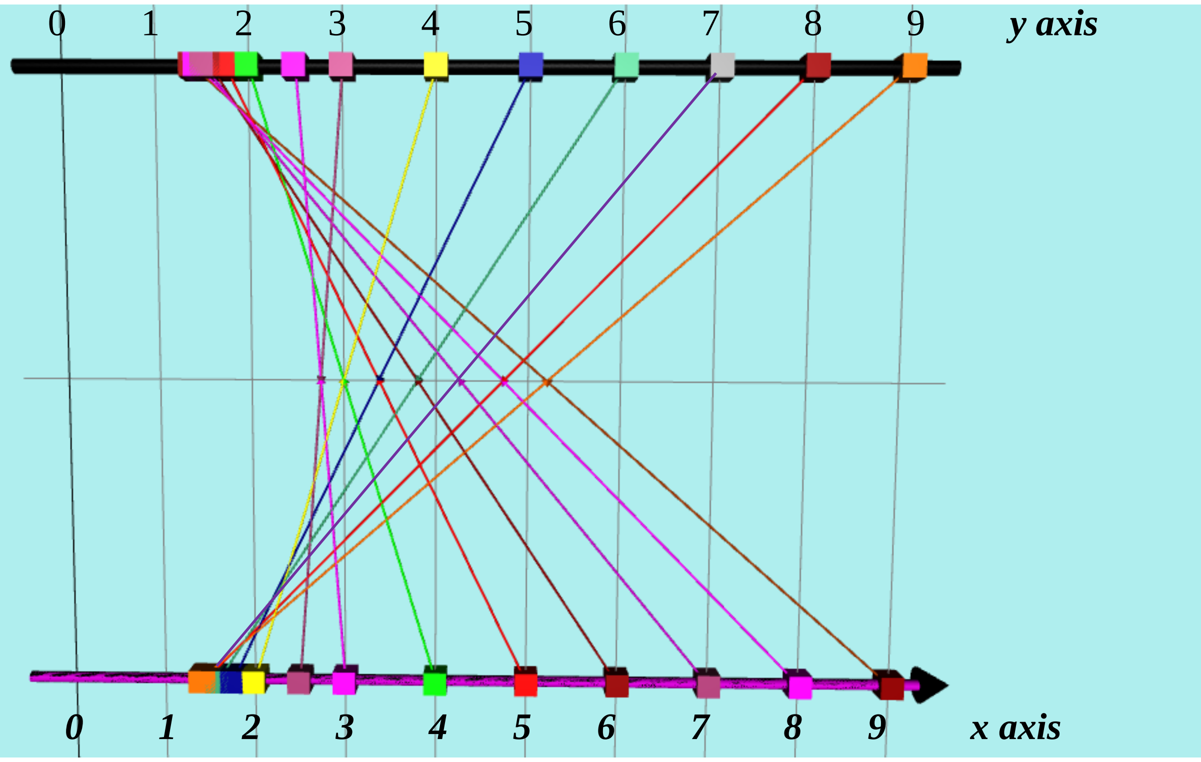
This is x = -1 + 2i
mapping onto y = -1 + 2i

This is x = 2 + 3i
mapping onto y = 2 + 3i

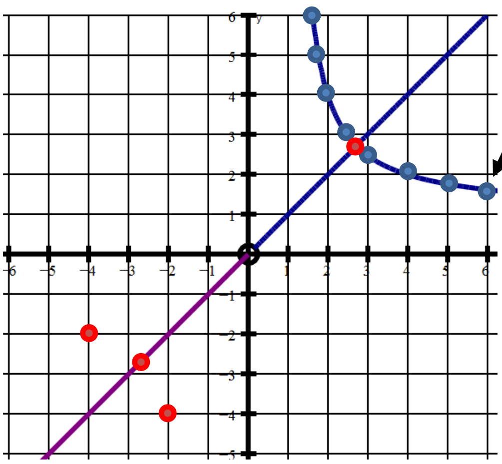
The special REAL points referred to earlier, such as the pairs:

x = 2, y = 4 and x = 4, y = 2
x = 3, y = 2.48 and x = 2.48, y = 3
x = 5, y = 1.77 and x = 1.77, y = 5
x = 6, y = 1.62 ***and x = 1.62, y = 6***
x = 7, y = 1.53 ***and x = 1.53, y = 7***
x = 8, y = 1.46 ***and x = 1.46, y = 8***
x = 9, y = 1.41 ***and x = 1.41, y = 9***

…can also be placed on a mapping of an x axis to a y axis.



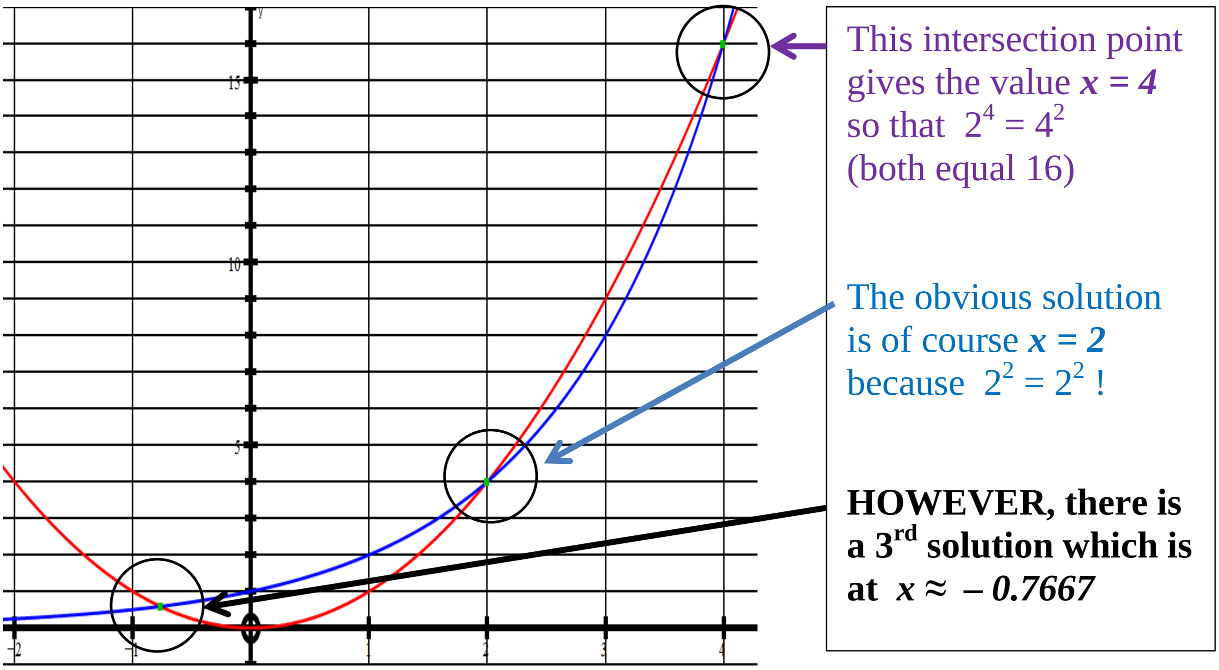
(This reminds me of the "curve stitching" that young children do!)



The above diagram represents the points which form this curved section resembling a hyperbola.

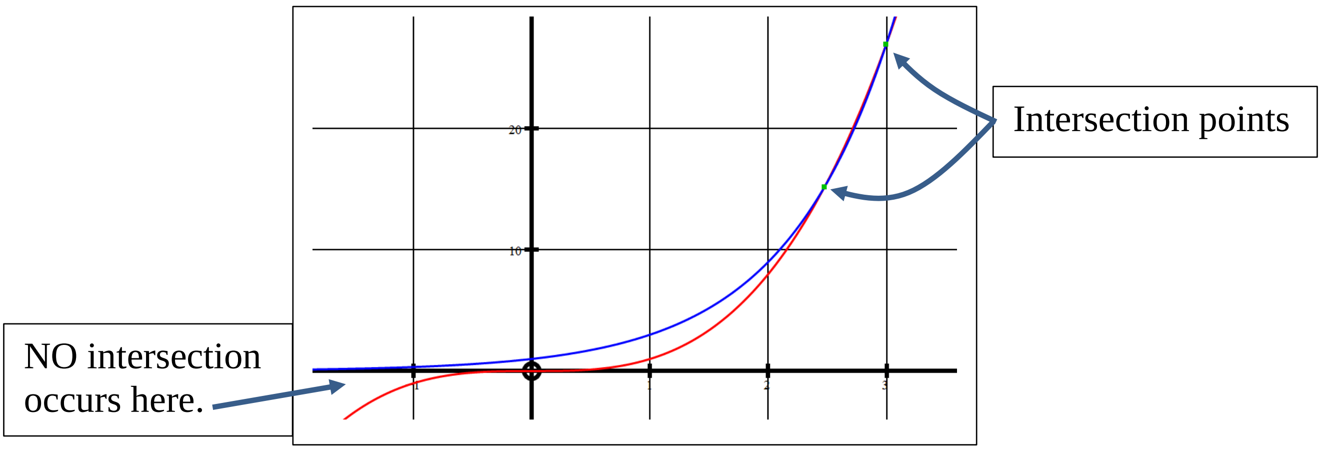
SOME MORE SPECIAL *REAL* POINTS!

Earlier, I referred to the "nice" whole number points (2, 4) and (4, 2) which fit the equation $y^x = x^y$.
Suppose we were not aware of these solutions and we say to ourselves,
"If y = 2, what would x be?"
ie Find x if $2^x = x^2$
If we think of this as the intersection of two graphs we could proceed as follows:
Draw $Y = 2^x$ ***and*** $Y = x^2$ ***(I am using a capital Y because these Y values are not the same as the y values in the equation!)***



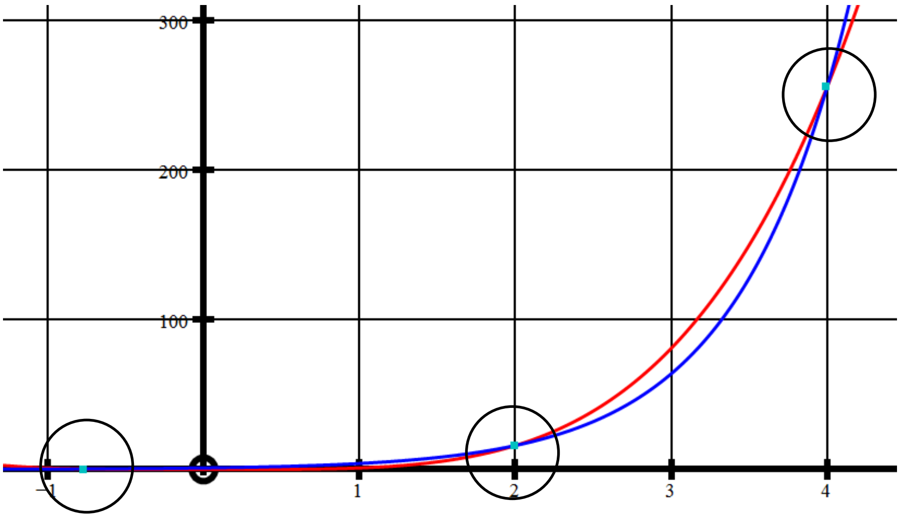
If we test this 3rd solution we get $2^{(-0.7667)} = 0.5878$
and $(-0.7667)^2 = 0.5878$

This method will not produce 3 solutions for ODD y values such as $y = 3$ because the graphs $Y = 3^x$ and $Y = x^3$ only intersect TWICE.



We will only get solutions for EVEN y values 4^x , 6^x , 8^x …

If we draw $Y = 4^x$ and $Y = x^4$ we get graphs which intersect 3 times.



The x values at the intersection points are x = 4, 2 and – 0.7667 (again)

CHECK:

$$4^{(-0.7667)} = 0.3455$$
$$(-0.7667)^4 = 0.3455$$

> *We can have: x = 4, y = – 0.7667*
> *AND x = – 0.7667, y = 4*

If we draw $Y = 6^x$ and $Y = x^6$ we also get graphs which intersect 3 times.
The x values are x = 6, 1.624 and – 0.7899

CHECK:

$$6^{(-0.7899)} = 0.2429$$
$$(-0.7899)^6 = 0.2429$$

> *We can have: x = 6, y = – 0.7899*
> *and x = – 0.7899, y = 6*

If we draw $Y = 8^x$ and $Y = x^8$ we also get graphs which intersect 3 times.
The x values are x = 8, 1.463 and – 0.8101

CHECK:

$$8^{(-0.8101)} = 0.1855$$
$$(-0.8101)^8 = 0.1855$$

> *We can have: x = 8, y = – 0.8101*
> *and x = – 0.8101, y = 8*

If we draw $Y = 10^x$ and $Y = x^{10}$ we also get graphs which intersect 3 times.
The x values are x = 10, 1.371 and – 0.8267

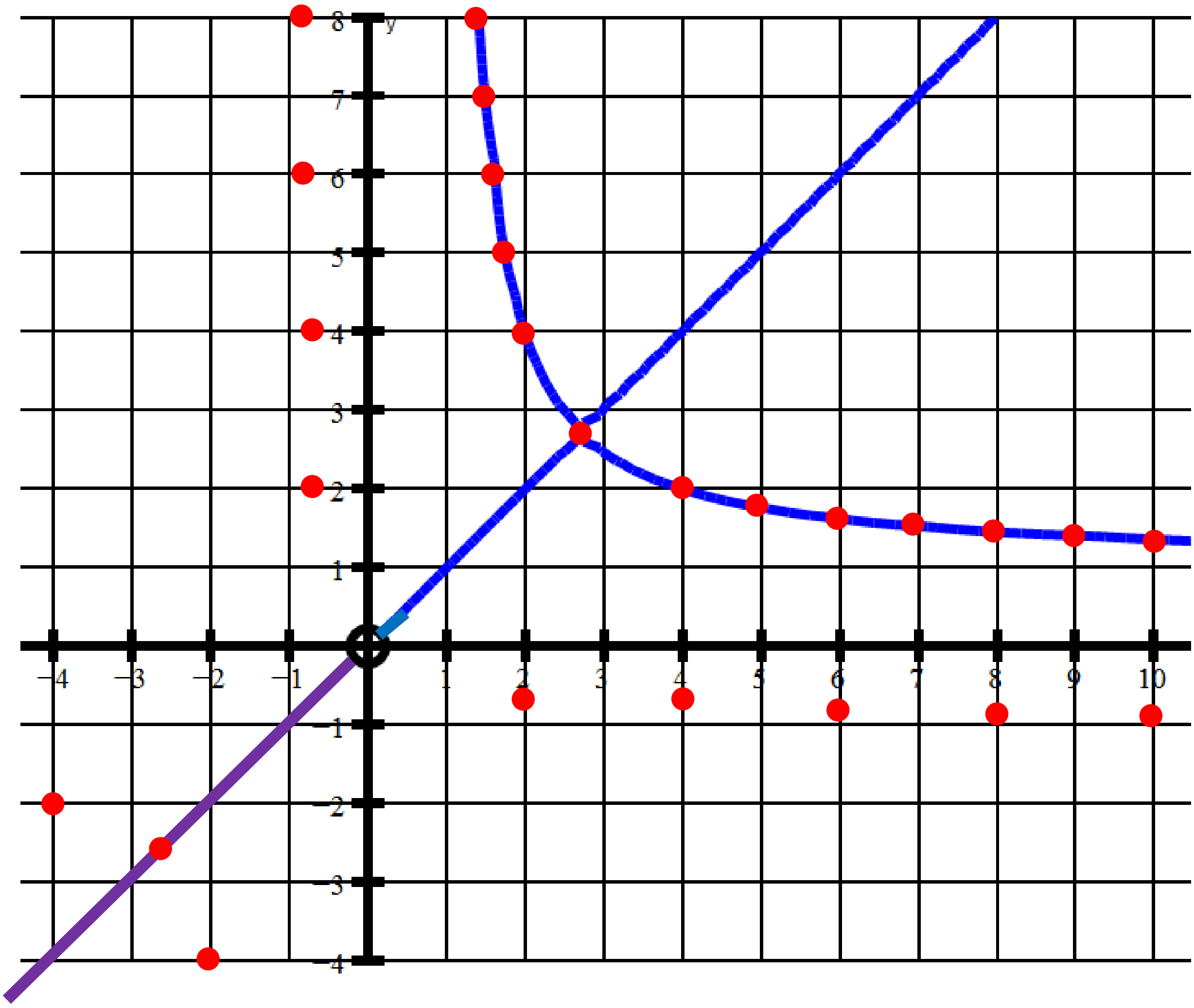
CHECK:

$$10^{(-0.8267)} = 0.1490$$
$$(-0.8267)^{10} = 0.1490$$

> *We can have: x = 10, y = – 0.8267*
> *and x = – 0.8267, y = 10*

These are all the solutions of $y^x = x^y$ I have found so far!

See below:



I have not YET found any MORE!!!

Apart from the infinite complex solutions of the form x = y = a + ib there are no actual "PHANTOM CURVES" because phantom graphs require A COMPLEX PLANE AND A REAL AXIS.

If the equation $y^x = x^y$ had any complex solutions such as x = a + bi and y = c + id then we would need a complex x plane and a complex y plane which would require 4 dimensional space. However, there may be some values a, b, c, d such that $(a + ib)^{(c + id)} = (c + id)^{(a + ib)}$ but I am still looking!